

**Lifted Cycle Inequalities
for the Asymmetric
Traveling Salesman Problem**

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Abstract

We investigate the family of facet defining inequalities for the asymmetric traveling salesman (ATS) polytope obtainable by lifting the cycle inequalities. We establish several properties of this family that earmark it as the most important among the asymmetric inequalities for the ATS polytope known to date: (i) The family is shown to contain members of unbounded Chvatal rank, whereas most known asymmetric inequalities are of Chvatal rank 1. (ii) For large classes within the family a coefficient pattern is identified that makes it easy to develop efficient separation routines. (iii) Each member of the family is shown to have a counterpart for the symmetric TS (STS) polytope that is often new, and is obtainable by mapping the inequality for the ATS polytope into a certain face of the STS polytope and then lifting the resulting inequality into one for the STS polytope itself.

1 Introduction

The symmetric traveling salesman (STS) polytope has been intensively studied for the last 20 years, with the result that a large variety of facet families are known (see [13] for a recent survey of the STS problem). The asymmetric traveling salesman (ATS) polytope has been studied mainly in the last few years. Each facet defining inequality for the STS polytope gives rise to a symmetric valid inequality for the ATS polytope, which in most cases is facet defining (see [5, 7, 13]). In addition, several families of asymmetric facet defining inequalities for the ATS polytope have been identified [1, 2, 3, 8], which have no obvious counterpart for the STS polytope. One of these families is that of lifted cycle inequalities, introduced and shown to be facet defining for the monotone ATS polytope by Grötschel and Padberg [9, 10], and for the ATS polytope itself (for cycles C with $|C| \leq n - 3$) by Balas and Fischetti [5].

In this paper we investigate the family of lifted cycle inequalities for the ATS polytope, and establish several properties that clearly earmark it as the most important among the families of asymmetric facet inducing inequalities known to date:

- (i) The family is shown to contain members of unbounded Chvátal rank. Most known families of asymmetric inequalities have Chvátal rank 1.
- (ii) For large classes within the family, a coefficient pattern is identified that makes it easy to develop efficient separation routines.
- (iii) Each member of the family is shown to have as a counterpart a facet defining inequality for the STS polytope, sometimes of a known type, sometimes new, obtainable by mapping the inequality for the ATS polytope into a certain face of the STS polytope and then lifting the resulting inequality into one for the STS polytope itself.

From a broader perspective, the importance of asymmetric facet defining inequalities is twofold: On the one hand, they make it advantageous to solve ATS problems directly as ATS problems

rather than reducing them to an STS problem; on the other, as we show in this paper, they have polyhedral implications for the STS problem. It is well known (see Karp [14], and more recently Jünger, Reinelt and Rinaldi [13]) that every ATS problem on the complete digraph $G = (N, A)$ can be restated as a STS problem defined on a special undirected graph with $2|N|$ vertices and $|A| + |N|$ edges, where $|N|$ specified edges must be contained in every tour. When solving the ATS problem as an STS problem, one can of course use the whole arsenal of separation routines developed for that problem. However, all the facet defining inequalities for the STS problem, along with the corresponding separation routines, have their direct counterparts for the ATS problem; and if the ATS problem is solved directly as such, then in addition one can also use the asymmetric facet defining inequalities and their separation routines, which are often simpler than their counterparts for the STS polytope. On the other hand, as pointed out in (iii) above and in more detail in section 8, the asymmetric facets of the ATS polytope can be used to derive new facets for the STS polytope. The polyhedral implications of the transformation of an ATS problem into an STS problem are, to our knowledge, examined here for the first time.

Let $G = (N, A)$ be the complete directed graph with $|N| = n$ nodes, and let P be the ATS polytope, i.e. the convex hull of all incidence vectors of tours (Hamiltonian dicycles) in G . Throughout this paper, $x(H) := \sum \{x_{ij} : (i, j) \in H\}$ for all $H \subseteq A$ and $x(S, T) := \sum \{x_{ij} : i \in S, j \in T\}$ for all $S, T \subset N$. As usual, we denote

$$\delta^+(S) := \{(i, j) \in A : i \in S, j \in N \setminus S\}$$

$$\delta^-(S) := \{(i, j) \in A : j \in S, i \in N \setminus S\}$$

$$\gamma(S) := \{(i, j) \in A : i, j \in S\}.$$

Then the ATS polytope can be defined as the convex hull of all 0-1 points in \mathbb{R}^A satisfying the *degree equations*

$$\begin{aligned} x(\delta^+(i)) &= 1, & i \in N \\ x(\delta^-(i)) &= 1, & i \in N \end{aligned}$$

and the *subtour elimination inequalities*

$$x(\gamma(S)) \leq |S| - 1, \quad S \subset N, \quad 2 \leq |S| \leq n - 1.$$

The set of all nonnegative points (possibly fractional) satisfying the degree equations and the subtour elimination constraints is called the *subtour elimination polytope*.

We will denote by \tilde{P} the monotone ATS polytope, defined as the convex hull of all 0-1 points satisfying the *degree inequalities*

$$\begin{aligned} x^+(\delta(i)) &\leq 1, & i \in N \\ x(\delta^-(i)) &\leq 1, & i \in N \end{aligned}$$

and the subtour elimination inequalities. \tilde{P} is closely related to the submissive of P , defined as the convex hull of all subsets of tours; the two notions coincide for complete graphs and many other cases (see [5] for a discussion of their relationship).

Unless otherwise stated, all directed cycles considered in this paper are simple. Let $S \subset N$, $S = \{i_1, i_2, \dots, i_s\}$, and let $C := \{(i_1, i_2), (i_2, i_3), \dots, (i_{s-1}, i_s), (i_s, i_1)\}$ be a directed cycle visiting all the nodes in S . For the sake of simplicity, we will use i_{j+1} and i_{j-1} to denote the successor and the predecessor, respectively, of node i_j in the cycle (hence $i_{s+1} \equiv i_1$ and $i_0 \equiv i_s$). A *chord* of C is an arc $(i_h, i_k) \in A$ such that $i_k \neq i_{h+1}$. Let R denote the set of chords of C . For every subset $F \subseteq A$, let $\tilde{P}(F) := \{x \in \tilde{P} : x_a = 0 \forall a \in F\}$. It is well known that

$$x(C) \leq |C| - 1$$

defines a (nontrivial) facet of the polytope $\tilde{P}(R)$. Moreover, let $R = \{a_1, a_2, \dots, a_m\}$; then the *lifted cycle inequality*

$$\alpha x := x(C) + \sum_{i=1}^m \alpha_{a_i} x_{a_i} \leq \alpha_0 := |C| - 1$$

defines a facet of \tilde{P} (see Grötschel and Padberg [9, 10]), where the lifting coefficients α_{a_i} ($i = 1, \dots, m$) are sequentially computed as the maximum value such that inequality $\alpha x \leq \alpha_0$ is valid for $\tilde{P}(\{a_{i+1}, \dots, a_m\})$. It is well known that (a) different sequences $\{a_i; i = 1, \dots, m\}$ may lead to different inequalities $\alpha x \leq \alpha_0$, and (b) the value of a given coefficient is largest if lifted first, and is

a monotone nonincreasing function of its position in the lifting sequence (with the position of the other coefficients kept fixed). In the case of lifted cycle inequalities $\alpha x \leq \alpha_0$, it is easily seen that $\alpha_{a_i} \in \{0, 1, 2\}$ for $i = 1, \dots, m$.

We will study the lifted cycle inequalities on \tilde{P} rather than P , since \tilde{P} is full dimensional, and the following result obtains:

Theorem 1.1 (*Balas and Fischetti [5]*) *Any lifted cycle inequality for \tilde{P} whose defining cycle has at most $n - 3$ arcs, induces a facet of P .*

Our paper is organized as follows: Section 2 characterizes those chord sets whose members may all have a coefficient equal to 2 in some lifted cycle inequality. Also, this section identifies those chords whose coefficients are forced to 0 by assigning value 2 to some other chord. Section 3 introduces the class of maximally 2-lifted cycle inequalities, which lend themselves to an easy derivation. The next two sections deal with maximally 2-lifted inequalities of rank 1 and of unbounded rank, respectively. Section 6 addresses the separation problem for the class of inequalities under discussion. Finally, Section 7 deals with the intriguing potential for deriving new facet defining inequalities for the symmetric TSP from the above classes of inequalities for the ATSP.

2 Two-Liftable Chord Sets

In this section we characterize those sets of chords that can get a coefficient 2 in the lifting process. We first note that a given lifted cycle inequality $\alpha x \leq \alpha_0$ obtained, say, via the chord sequence a_1, \dots, a_m , can always be obtained via an equivalent chord sequence in which all the chords with coefficient 2 appear (in any order) at the beginning of the sequence, while all the chords with coefficient 0 appear (in any order) at the end of the sequence. We call any such chord sequence *canonical*. Indeed, consider swapping the positions in the lifting sequence of two consecutive chords. Clearly, either (a) their coefficients remain unchanged, or (b) the coefficient of the chord moved to the left increases and the other one decreases. Case (b) cannot occur when a chord with coefficient

2 is moved to the left or a chord with coefficient 0 is moved to the right. Therefore a sequence of swaps of the above type leads to the desired canonical form.

A given set $H \subset R$ is called *2-liftable* if there exists a chord sequence producing a lifted cycle inequality $\alpha x \leq \alpha_0$ such that $\alpha_a = 2$ for all $a \in H$. Because of the above property, such a sequence can w.l.o.g. be assumed to be canonical. It then follows that H is 2-liftable if and only if $\alpha x := x(C) + 2x(H) \leq |C| - 1$ is a valid inequality for $\tilde{P}(R \setminus H)$. We will show that in order to impose this condition it is necessary and sufficient to forbid the presence in H of certain patterns of chords.

Two distinct arcs (i, j) and (u, v) are called *compatible* when there exists a point in \tilde{P} containing both, i.e. when $i \neq u$, $j \neq v$, and (i, j) and (u, v) do not form a 2-cycle. Two arcs that are not compatible are called *incompatible*. Given a chord (i_a, i_b) of C , we call *internal* w.r.t. (i_a, i_b) the nodes i_b, i_{b+1}, \dots, i_a ; *external* the nodes i_{a+1}, \dots, i_{b-1} . Given two compatible chords (i_a, i_b) and (i_c, i_d) , we say that (i_c, i_d) *crosses* (i_a, i_b) if nodes i_c and i_d are not both internal or both external w.r.t. (i_a, i_b) ; see Figure 1 for an illustration. Note that (i_c, i_d) crosses (i_a, i_b) if and only if (i_a, i_b) crosses (i_c, i_d) , i.e., the property is symmetric.

We define a *noose* in $C \cup R$ as a simple alternating (in direction) cycle $Q := \{a_1, b_1, a_2, b_2, \dots, a_q, b_q\}$ of $2q \geq 4$ distinct arcs $a_i \in R$ and $b_i \in C$ ($i = 1, \dots, q$), in which all adjacent arcs in the sequence (including b_q and a_1) are incompatible and all chords are pairwise noncrossing (see Figure 2).

Theorem 2.1 *A chord set $H \subset R$ is 2-liftable if and only if $C \cup H$ contains no pair of crossing chords and no noose.*

Proof. We show that for any cycle C and any chord set H , the inequality

$$\alpha x := x(C) + 2x(H) \leq |C| - 1$$

is valid for $\tilde{P}(R \setminus H)$ if and only if the condition of the theorem is satisfied. In particular, we show that if H contains a pair of crossing chords or a noose, one can always exhibit a point in $\tilde{P}(R \setminus H)$

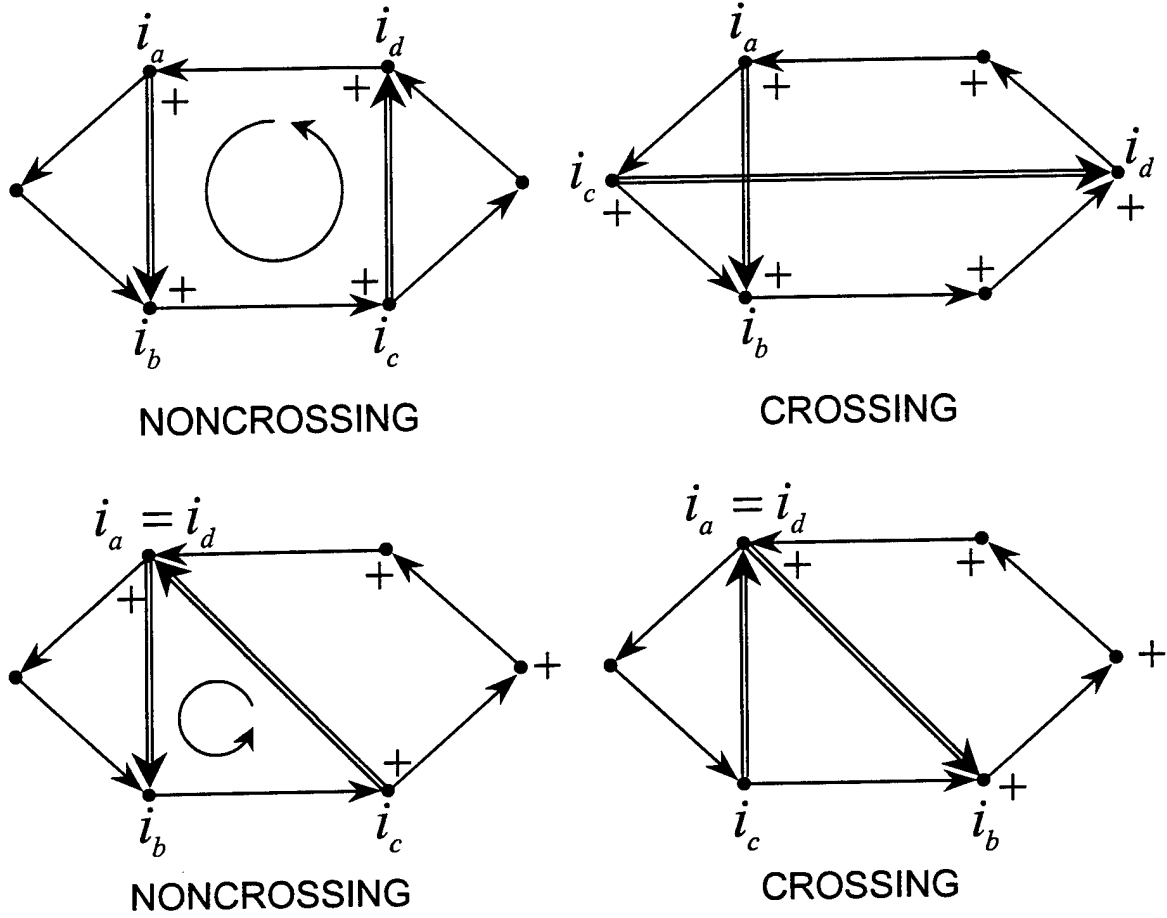


Figure 1: Crossing and noncrossing chords. The internal nodes w.r.t. (i_a, i_b) are marked with $+$, and the chords are drawn in double lines.

that violates $\alpha x \leq |C| - 1$. Conversely, we prove by induction on $|H|$ that the absence of crossing pairs of chords and nooses implies the validity of $\alpha x \leq |C| - 1$ for $\tilde{P}(R \setminus H)$.

(a) Necessity. Suppose H contains a pair of crossing chords, say (i_a, i_b) and (i_c, i_d) . Then (possibly after interchanging the roles of the two chords) the situation is that depicted in Figure 3(a), where case $i_a = i_d$ (but not $i_c = i_b$) is allowed. The arc set

$$F := C \cup \{(i_a, i_b), (i_c, i_d)\} \setminus \{(i_a, i_{a+1}), (i_c, i_{c+1}), (i_{b-1}, i_b), (i_{d-1}, i_d)\}$$

contains at least $|C| - 4$ members of C (note that (i_c, i_{c+1}) and (i_{b-1}, i_b) may be the same), plus the two chords (i_a, i_b) and (i_c, i_d) . Then clearly the characteristic vector x^* of F belongs to $\tilde{P}(R \setminus H)$ and violates the inequality $\alpha x \leq |C| - 1$.

Suppose now that $C \cup H$ contains a noose $Q = \{a_1, b_1, a_2, b_2, \dots, a_q, b_q\}$, and let $\beta_i \in C$ be the

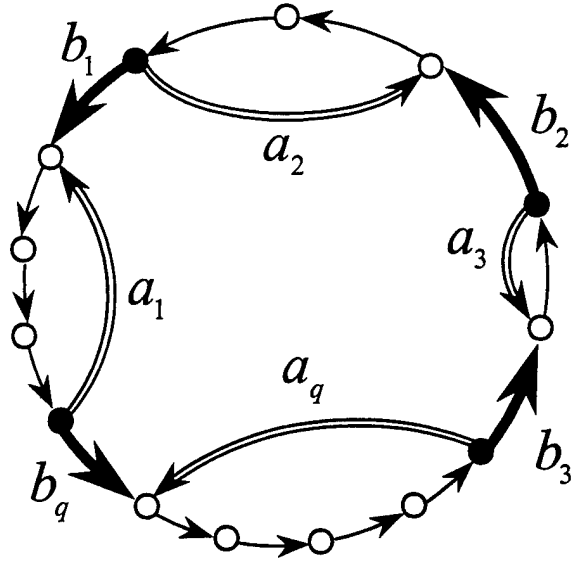


Figure 2: A noose Q with $q = 4$.

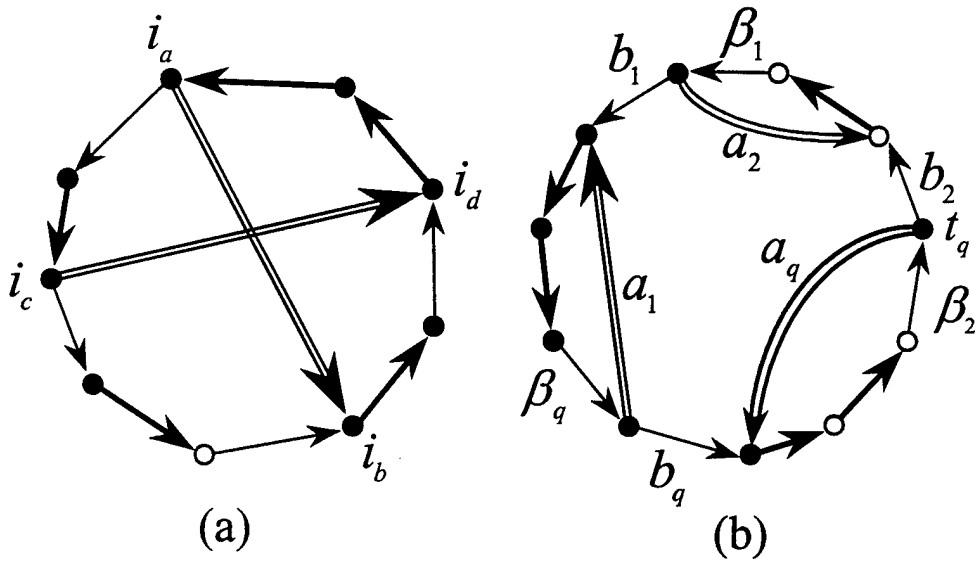


Figure 3: Illustration of the constructions used in the proof of Theorem 2.1 (in bold and double line the arcs in F)

arc preceding b_i in C ($i = 1, \dots, q$); see Figure 3(b). Define

$$F := C \cup \{a_1, a_2, \dots, a_q\} \setminus \{b_1, \beta_1, b_2, \beta_2, \dots, b_q, \beta_q\}.$$

It can easily be seen that the characteristic vector x^* of F belongs to $\tilde{P}(R \setminus H)$, and satisfies $\alpha x^* = |C| + 2q - 2q = |C|$, hence $\alpha x \leq |C| - 1$ is not valid for $\tilde{P}(R \setminus H)$.

(b) Sufficiency. We prove the claim by induction on $|H|$. When $|H| = 0$ the claim trivially holds. Now assume that it holds for a cycle C' and all chord sets H' with $|H'| \leq \mu$, and consider any given cycle C and any chord subset H , where $|H| = \mu + 1$, satisfying the condition of the theorem. We have to show that, given any vertex x^* of $\tilde{P}(R \setminus H)$, $\alpha x^* \leq |C| - 1$ holds.

Let $F := \{a \in C \cup H : x_a^* = 1\}$, and note that $\alpha x^* = |F \cap C| + 2|F \cap H|$. If there exists $a^* \in H$ such that $x_{a^*}^* = 0$, then clearly $\alpha x^* = x^*(C) + 2x^*(H \setminus \{a^*\})$ and the claim follows directly from the induction hypothesis. Hence assume $x_a^* = 1$ for all $a \in H$. This implies that the chords in H are pairwise compatible. Now let (i_a, i_b) be any chord in H , and define the set $I := \{i_b, i_{b+1}, \dots, i_a\}$ and $E := \{i_{a+1}, i_{a+2}, \dots, i_{b-1}\}$ of the associated internal and external nodes, respectively. Since $(i_a, i_b) \in F$, we must have $(i_{b-1}, i_b) \notin F$ and $(i_a, i_{a+1}) \notin F$. In addition, no chord in H crosses (i_a, i_b) by assumption, hence F can be partitioned into $F_I \cup F_E$, where $F_I \subset I \times I$ and $F_E \subset E \times E$. Now let α_I^* and x_I^* (resp., α_E and x_E^*) denote the restriction of α and x^* to $I \times I$ (resp., $E \times E$). Then $\alpha x^* = \alpha_I x_I^* + \alpha_E x_E^*$. We will show that $\alpha_I x_I^* \leq |I|$ and $\alpha_E x_E^* \leq |E| - 1$, from which $\alpha x^* \leq |C| - 1$ and hence the claim follows.

Consider first the term $\alpha_I x_I^* = |F_I \cap C| + 2|F_I \cap H|$. Define a new cycle $C_I := \{(i_a, i_b), (i_b, i_{b+1}), \dots, (i_{a-1}, i_a)\}$, and a chord set $H_I := H \cap (I \times I) \setminus \{(i_a, i_b)\}$. It can easily be seen that $C_I \cup H_I$ contains no crossing chords and no nooses (note that a noose in $C_I \cup H_I$ would be a noose in $C \cup H$ as well). Hence the induction hypothesis implies that $x^*(C_I) + 2x^*(H_I) \leq |C_I| - 1 = |I| - 1$, where the left-hand side equals $\alpha_I x_I^* - 1$ (since (i_a, i_b) is a chord of C but not of C_I).

A similar reasoning applies to $\alpha_E x_E^* = |F_E \cap C| + 2|F_E \cap H|$. Indeed, define a new cycle $C_E := \{(i_{b-1}, i_{a+1}), (i_{a+1}, i_{a+2}), \dots, (i_{b-2}, i_{b-1})\}$ and a chord set $H_E := H \cap (E \times E)$. Note that

$(i_{b-1}, i_{a+1}) \notin H$, since otherwise it, together with (i_a, i_b) induces a noose. Clearly H_E contains no crossing chords, since H does not. As to nooses, it suffices to observe that any noose Q_E in $C_E \cup H_E$ would correspond to a noose Q in $C \cup H$, where $Q := Q_E$ if $(i_{b-1}, i_{a+1}) \notin Q_E$, and $Q := (Q_E \setminus \{(i_{b-1}, i_{a+1})\}) \cup \{(i_{b-1}, i_b), (i_a, i_b), (i_a, i_{a+1})\}$ otherwise. Then the induction hypothesis leads to $\alpha_E x_E^* = x^*(C_E) + 2x^*(H_E) \leq |C_E| - 1$. This completes the proof. \square

An immediate consequence of Theorem 2.1 is that the set of 2-liftable chords is never empty for $|C| \geq 3$. It then follows that the subtour elimination inequalities, in which every chord has a coefficient 1, are not sequentially lifted inequalities (they can be obtained from the corresponding cycle inequality by simultaneous lifting). Any chord that is lifted first in a sequential lifting procedure must get a coefficient 2.

We might note at this point that we have touched upon an important point of difference between the symmetric and asymmetric TS polytopes. In the case of the STS polytope, the subtour elimination inequalities are the only kinds of lifted cycle inequalities: whichever chord is lifted first, it gets a coefficient of 1, and so does the chord that is lifted last.

Assigning a chord (i, j) the coefficient $\alpha_{ij} = 2$ forces to 0 the coefficients of several other chords.

Theorem 2.2 *Let (i_a, i_b) be a chord of C such that $\alpha_{i_a i_b} = 2$. Then the following chords must have coefficient 0 (see Figure 4):*

(i) (i_j, i_{a+1}) for all $j = b, b+1, \dots, a-1$;

(ii) (i_{b-1}, i_ℓ) for all $\ell = b+1, b+2, \dots, a$.

Proof. For any chord satisfying (i) or (ii) that is assigned a coefficient greater than 0 we exhibit a feasible point x that violates $\alpha x \leq |C| - 1$.

In case (i), such a point is given by the arc set $C \setminus \{(i_a, i_{a+1}), (i_{b-1}, i_b), (i_j, i_{j+1})\} \cup \{(i_a, i_b), (i_j, i_{a+1})\}$; in case (ii), the corresponding arc set is $C \setminus \{(i_a, i_{a+1}), (i_{b-1}, i_b), (i_{\ell-1}, i_\ell) \cup \{(i_a, i_b), (i_{b-1}, i_\ell)\}$. \square

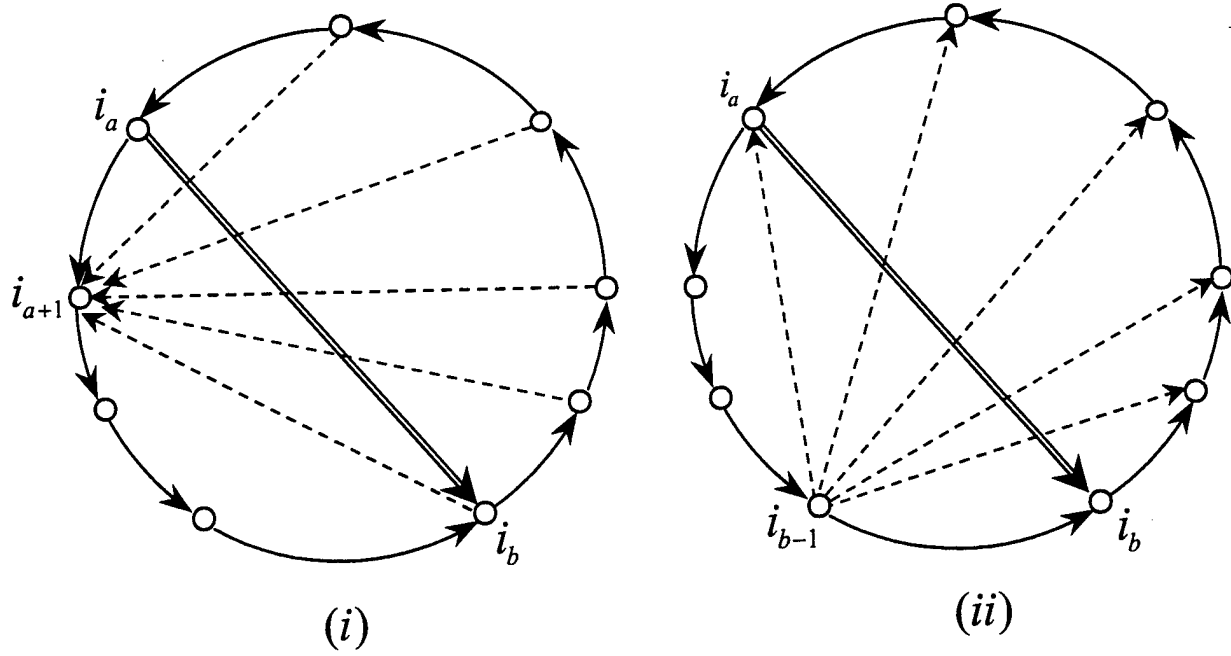


Figure 4: The dashed lines represent chords with coefficient 0.

Corollary 2.3 *If the chord (i_a, i_{a+2}) has coefficient 2, then all chords incident with node i_{a+1} must have coefficient 0.*

Proof. It is enough to observe that chords (i_{a+1}, i_a) and (i_{a+2}, i_{a+1}) , which are not covered by Theorem 2.2, cannot be assigned a coefficient greater than 0 due to the feasibility of the points associated with arcsets $C \setminus \{(i_a, i_{a+1}), (i_{a+1}, i_{a+2}), (i_{a-1}, i_a)\} \cup \{(i_a, i_{a+2}), (i_{a+1}, i_a)\}$ and $C \setminus \{(i_a, i_{a+1}), (i_{a+1}, i_{a+2}), (i_{a+2}, i_{a+3}), \dots\} \cup \{(i_a, i_{a+2}), (i_{a+2}, i_{a+1})\}$ \square

3 Maximally 2-Lifted Cycle Inequalities

Given a lifted cycle inequality with a given set of 2-lifted chords and a corresponding set of chords with coefficients forced to 0, the size of the remaining coefficients depends in general on the lifting sequence. Next we characterize a class of lifted cycle inequalities whose 0-1 coefficients are largely sequence-independent.

A set Q_2 of 2-liftable chords is termed *maximal* if no set of the form $Q_2 \cup \{(i, j)\}$, $(i, j) \in R \setminus Q_2$, is 2-liftable. A lifted cycle inequality whose set of chords with coefficient 2 is maximal 2-liftable,

will be called a *maximally 2-lifted* cycle inequality.

Proposition 3.1 *The maximum cardinality of a 2-liftable chord set is $|C| - 2$.*

Proof. We show by induction on $|C|$ that any set of pairwise noncrossing chords has at most $|C| - 2$ elements. On the other hand, examples of lifted cycle inequalities with $|C| - 2$ 2-liftable chords will be exhibited in the following sections.

The statement is clearly true for $|C| = 3$. Suppose it is true for $|C| = 3, 4, \dots, k - 1$, and let C be any cycle with $|C| = k \geq 4$. For any chord (i, j) , we denote by C_{ij} the shorter of the two cycles contained in $C \cup \{(i, j)\}$. Consider any set H of pairwise noncrossing chords of C , and let (i_a, i_b) be a member of H for which $|C_{i_a i_b}|$ is the largest. W.l.o.g., let $a, a + 1, \dots, b$ be denoted $1, 2, \dots, q$. Then C has $q - 2$ external nodes relative to $(i_a, i_b) \equiv (i_1, i_q)$. Further, by the definition of (i_1, i_q) , every chord in H incident with an external node is of the form (i_j, i_ℓ) with $j > \ell$. Since H contains no crossing chords, no chord incident with an external node is incident with either i_1 or i_q . Therefore, the number of such chords is at most $q - 3$. On the other hand, all remaining chords except for (i_1, i_q) are also noncrossing chords of $C_{i_1 i_q}$, and by the induction hypothesis their number is at most $|C_{i_1 i_q}| - 2 = |C| - (q - 2) - 2 = |C| - q$. It then follows that $|H| \leq (q - 3) + (|C| - q) + 1 = |C| - 2$, as claimed. This completes the induction. \square

Notice that not all sequentially lifted cycle inequalities are maximally 2-lifted. Examples of (facet inducing) lifted cycle inequalities with a single 2-lifted chord are shown in Figure 5. (For the first graph of that figure, the appropriate lifting sequence is $(3, 2), (1, 4), (3, 1), (4, 2)$ etc.). In fact, the h -canonical form (see [4] for a definition) of any CAT inequality (where h is a source) is a lifted cycle inequality with a single 2-lifted chord.

On the other hand, we can identify some large classes of maximally 2-liftable cycle inequalities with nice properties that are useful in the context of separation. The next theorem gives a sufficient condition for a given inequality related to a cycle C to be a (facet defining) lifted cycle inequality for \tilde{P} .

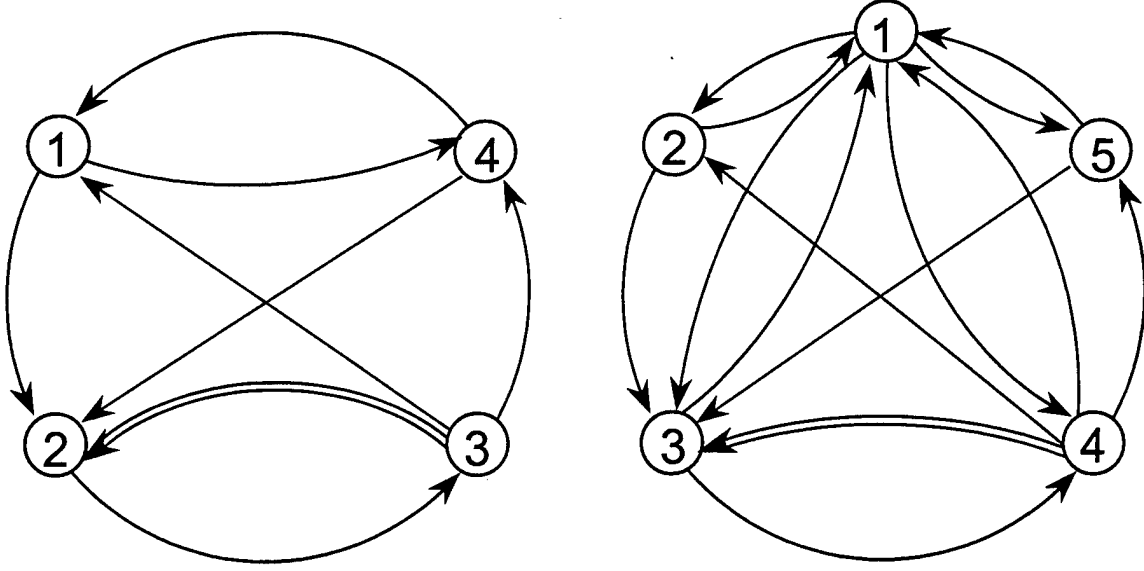


Figure 5: Support graphs of two lifted cycle inequalities having a single chord with coefficient 2.

Theorem 3.2 Let $\alpha x \leq \alpha_0$ be an inequality with $\alpha_{ij} = 1$ for all arcs (i, j) of a given cycle C of length $|C| = \alpha_0 + 1 \leq n - 1$, where $\alpha_{ij} \in \{0, 1, 2\}$ for all chords (i, j) of C , and $\alpha_{ij} = 0$ for all other arcs. Let $Q_t := \{(i, j) \in R : \alpha_{ij} = t\}$ for $t = 0, 1, 2$, and assume the following conditions hold:

- (a) Q_2 is a maximal 2-liftable set;
- (b) all 0 coefficients are maximal, i.e. $\alpha_{ij} = 0$ implies the existence of $x^* \in \tilde{P}$ such that $\alpha x^* = \alpha_0$ and $x_{ij}^* = 1$;
- (c) the inequality $\alpha x \leq \alpha_0$ is valid for \tilde{P} .

Then $\alpha x \leq \alpha_0$ is a maximally 2-lifted cycle inequality, hence facet defining for \tilde{P} .

Proof. If (a), (b) and (c) hold, then lifting Q_2 first and then the remaining chords in any order in which the chords in Q_1 precede those in Q_0 yields $\alpha x \leq \alpha_0$ \square

A particularly friendly (and, it turns out, rich) class of lifted cycle inequalities is that for which $Q_0 := \{(i, j) \in R : \alpha_{ij} = 0\}$ is just the set of chords whose coefficients are forced to 0 by the

conditions of Theorem 2.2 and Corollary 2.3. For this class condition (b) holds automatically, and the only condition to be checked is (c):

Corollary 3.3 *Let $\alpha x \leq \alpha_0$ and let $Q_t := \{(i, j) \in R : \alpha_{ij} = t\}$ for $t = 0, 1, 2$. If Q_2 is a maximal 2-liftable set and Q_0 is the set whose coefficients are forced to 0 by the conditions of Theorem 2.2 and Corollary 2.3, then $\alpha x \leq \alpha_0$ is a (facet defining) lifted cycle inequality for \tilde{P} if and only if it is valid for \tilde{P} .*

One familiar subclass of this class is that of the D_k^+ and D_k^- inequalities introduced by Grötschel and Padberg [9, 10]. Another subclass will be introduced in Section 6.

4 Maximally 2-Lifted Cycle Inequalities of Rank 1.

In this section we introduce two new classes of lifted cycle inequalities of Chvátal-rank 1. We start by pointing out that the pattern of 2-liftable chords in a rank 1 inequality has to satisfy an additional condition (besides those required for 2-liftability): it has to define a nested family of node sets in the following sense. A family \mathcal{F} of sets is *nested* if for every $S_1, S_2 \in \mathcal{F}$, either $S_1 \subseteq S_2$, or $S_2 \subseteq S_1$, or $S_1 \cap S_2 = \emptyset$. Given a cycle C and a chord (i_a, i_b) of C , we will denote again $C_{i_a i_b} := \{(i_a, i_b), (i_b, i_{b+1}), \dots, (i_{a-1}, i_a)\}$, i.e. $C_{i_a i_b}$ is the shorter of the two cycles contained in $C \cup \{(i_a, i_b)\}$. For any arc set F , we denote by $N(F)$ the set of nodes spanned by F .

Theorem 4.1 *Let $\alpha x \leq \alpha_0$ be a lifted cycle inequality with cycle C , chord set R and $Q_2 := \{(i, j) \in R : \alpha_{ij} = 2\}$. If $\alpha x \leq \alpha_0$ is of rank 1, then the node sets $N(C)$ and $N(C_{i_a i_b})$ for all $(i_a, i_b) \in Q_2$ form a nested family.*

Before we prove this theorem, we notice its implication that rank 1 lifted cycle inequalities form a very special subclass indeed: for every $(i_a, i_b) \in Q_2$, the path from i_b to i_a in C never meets the tail of any chord in Q_2 before meeting its head. This implies, among other things, that Q_2 cannot contain a 2-cycle.

For the proof of the theorem we need the following result, also useful for later proofs.

Lemma 4.2 *In any Chvátal derivation of a lifted cycle inequality $\alpha x \leq \alpha_0$, the following inequalities must appear with a positive multiplier:*

$$(a) \ x(\gamma(N(C))) \leq |C| - 1;$$

$$(b) \ x(\gamma(N(C_{i_a i_b}))) \leq |C_{i_a i_b}| - 1, \quad \text{for all } (i_a, i_b) \in Q_2;$$

$$(c) \ x(\delta^+(i_a)) \leq 1, \quad \text{for all } (i_a, i_b) \in Q_2;$$

$$(d) \ x(\delta^-(i_b)) \leq 1, \quad \text{for all } (i_a, i_b) \in Q_2.$$

Proof. Suppose there exists a Chvátal derivation of $\alpha x \leq \alpha_0$ in which inequality (a) above does not have a positive coefficient. Then $\alpha x \leq \alpha_0$ must be valid for the polytope \tilde{P}^* , defined as the convex hull of points $x \in \{0, 1\}^A$ satisfying the degree inequalities and all the subtour elimination inequalities except (a). But the point x^* defined by $x_e^* = 1$ for $e \in C$, $x_e^* = 0$ otherwise, belongs to \tilde{P}^* but violates $\alpha x \leq \alpha_0$; a contradiction. The same argument applies to any of the remaining inequalities listed above, with the following definitions of x^* : for (b), $x_e^* = 1$ if $e \in C \cup \{(i_a, i_b)\} \setminus \{(i_a, i_{a+1}), (i_{b-1}, i_b)\}$, $x_e^* = 0$ otherwise; for (c), $x_e^* = 1$ if $e \in C \cup \{(i_a, i_b)\} \setminus \{(i_{a-1}, i_a), (i_{b-1}, i_b)\}$, $x_e^* = 0$ otherwise; for (d), $x_e^* = 1$ if $e \in C \cup \{(i_a, i_b)\} \setminus \{(i_a, i_{a+1}), (i_b, i_{b+1})\}$, $x_e^* = 0$ otherwise. \square

Proof of Theorem 4.1. We will show that there always exists a Chvátal derivation of the rank 1 inequality $\alpha x \leq \alpha_0$, which uses only subtour elimination inequalities whose associated node sets form a nested family \mathcal{F} . In view of the above Lemma, each of the subtour elimination inequalities (a) and (b) must belong to \mathcal{F} , from which the claimed result follows.

Let $A_1 x \leq b_1$ and $A_2 x \leq b_2$ represent the degree inequalities and subtour elimination inequalities, respectively. Since $\alpha x \leq \alpha_0$ is of rank 1 and is facet defining for the full dimensional polytope \tilde{P} , with relatively prime integer coefficients, there exist vectors $u_1, u_2 \geq 0$ such that

$$\begin{aligned} u_1 A_1 + u_2 A_2 &\geq \alpha \\ u_1 b_1 + u_2 b_2 &< \alpha_0 + 1 \end{aligned}$$

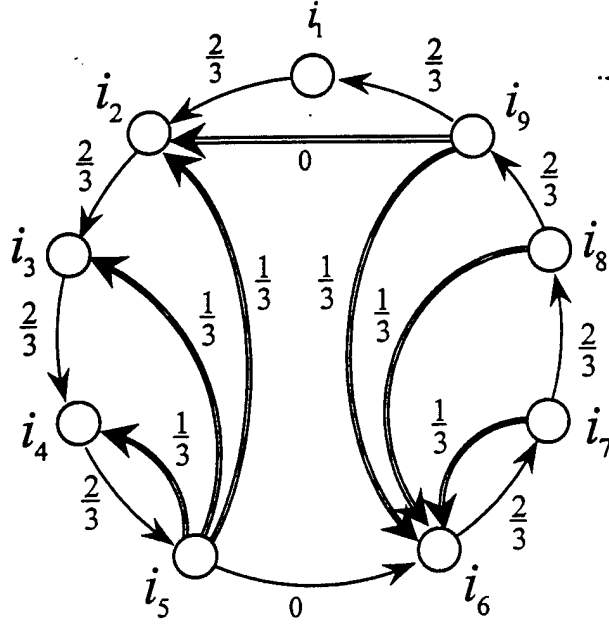


Figure 6: Counterexample for the converse of Theorem 4.1

(see Queyranne and Wang [15]).

Given the vector u_1 , an associated u_2 satisfying the above inequalities can be obtained by solving the linear program

$$\min\{u_2 b_2 : u_2 A_2 \geq \alpha - u_1 A_1, u_2 \geq 0\},$$

whose dual is

$$\max\{(\alpha - u_1 A_1)x : A_2 x \leq b_2, x \geq 0\}.$$

It is known from arborescence theory that the above minimization problem always has an optimal solution \bar{u}_2 whose support is associated with a noncrossing set of subtour elimination constraints. (For a proof of this result, based on an “uncrossing” argument, see for instance [17], page 313). \square

Theorem 4.1 gives a necessary condition for $\alpha x \leq \alpha_0$ to be of rank 1. Interestingly, this condition is not sufficient, as shown by the following example. Consider the cycle C and the 2-liftable set of chords of Figure 6. One can easily see that node sets $N(C)$ and $N(C_{i_a i_b})$ for $(i_a, i_b) \in Q_2$ form a nested family, as required in Theorem 4.1. However, no lifted cycle inequality $\alpha x \leq \alpha_0$ with the

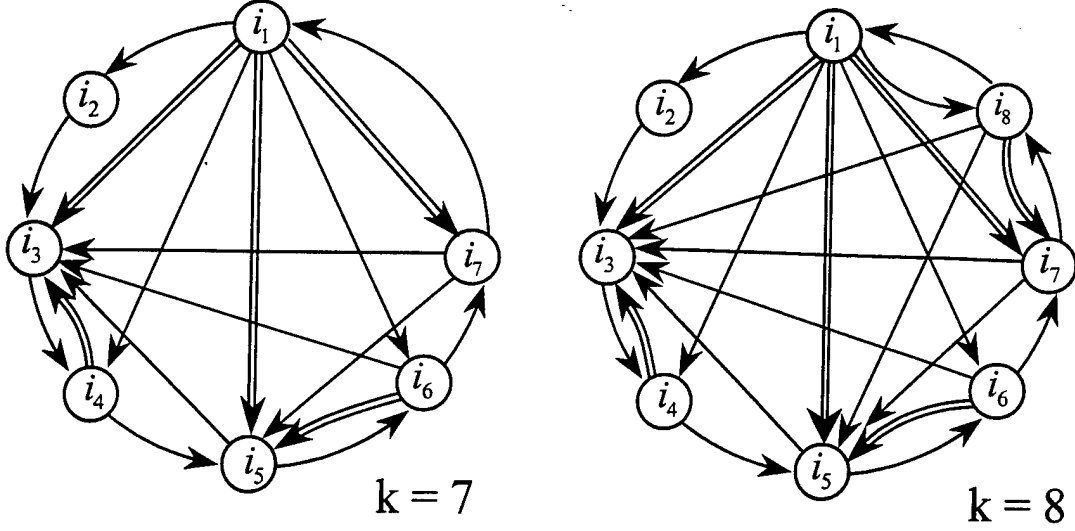


Figure 7: Support graphs of two shell inequalities.

2-chord pattern of Figure 6 can be of rank 1, as certified by the point x^* with $x_{ij}^* := \frac{2}{3}$ for all $(i, j) \in C \setminus \{(i_5, i_6)\}$, $x_{ij}^* := \frac{1}{3}$ for all $(i, j) \in Q_2 \setminus \{(i_9, i_2)\}$, $x_{ij}^* := 0$ for all other arcs. Indeed, x^* satisfies all degree and subtour elimination inequalities, but $\alpha x^* = \frac{28}{3} > \alpha_0 + 1 = 9$. This implies [16] that $\alpha x \leq \alpha_0$ cannot be of rank 1.

We now introduce two new large classes of maximally 2-lifted cycle inequalities of rank 1.

Theorem 4.3 *Let C be the cycle visiting in sequence the nodes i_1, \dots, i_k for some $4 \leq k \leq n-1$. Then the shell inequality*

$$\begin{aligned} \alpha x := & x(C) + \sum_{\substack{3 \leq j \leq k \\ j \text{ odd}}} x(\{i_{j+1}, \dots, i_k, i_1\}, i_j) \\ & + \sum_{j=3}^k x_{i_1 i_j} + \sum_{\substack{3 \leq j \leq k-1 \\ j \text{ odd}}} x_{i_{j+1} i_j} \leq k-1 =: \alpha_0 \end{aligned}$$

is a rank 1 maximally 2-lifted cycle inequality, hence facet defining for \tilde{P} . (See Figure 7.)

Proof. We show that the conditions of Theorem 3.2 are satisfied.

(a) Q_2 is easily seen to be a 2-liftable set (it contains no crossing arcs or nooses), and to consist of $|C| - 2$ chords, which from Proposition 3.1 implies that it is maximal.

(b) We claim that each 0 coefficient of a chord is maximal. Let (i_t, i_j) be any chord with $\alpha_{i_t i_j} = 0$. If this chord is incident with i_2 , then its coefficient is forced to 0 by the fact that $(i_1, i_3) \in Q_2$ (see Corollary 2.3). Otherwise there are two main cases.

Case 1: t is even ($t \geq 4$). If $j \in \{1\} \cup \{t+2, \dots, k\}$, then again $\alpha_{i_t i_j} = 0$ because $(i_1, i_{t+1}) \in Q_2$ (Theorem 2.1, (ii)). Otherwise we have $4 \leq j \leq t-2$, j even, and we prove the claim by exhibiting a point $x^* \in \tilde{P}$ such that $x_{i_t i_j}^* = 1$ and $\alpha x^* = \alpha_0$: namely, x^* is defined by $x_{uv}^* = 1$ for $(u, v) \in C \setminus \{(i_1, i_2), (i_{j-2}, i_{j-1}), (i_j, i_{j+1}), (i_t, i_{t+1})\} \cup \{(i_1, i_{j+1}), (i_j, i_{j-1}), (i_t, i_j)\}$, $x_{uv}^* = 0$ otherwise (notice that $\alpha_{i_1 i_{j+1}} = \alpha_{i_j i_{j-1}} = 2$, while $\alpha_{uv} = 1$ for the remaining arcs of the above set, except for (i_t, i_j)).

Case 2: t is odd ($t \geq 3$). We exhibit a point $x^* \in \tilde{P}$ such that $x_{i_t i_j}^* = 1$ and $\alpha x^* = \alpha_0$, which proves the claim. Let A^* denote the support of x^* . The arcs in $A^* \setminus \{(i_t, i_j)\}$ have coefficient 1, except for the starred ones, which have coefficient 2.

If $j \in \{t+2, \dots, k\}$, we set $A^* = \{(i_{j+1}, i_{j+2}), \dots, (i_k, i_1), (i_1, i_2), \dots, (i_{t-2}, i_{t-1})\} \cup \{(i_t, i_j)\} \cup \{(i_{t+1}, i_t)^*, (i_{t+3}, i_{t+2})^*, \dots, (i_j, i_{j-1})^*\}$ if j is even; and we set $A^* = \{(i_t, i_j), (i_j, i_{j+1}), \dots, (i_k, i_1), (i_1, i_2), \dots, (i_{t-2}, i_{t-1})\} \cup \{(i_{t+1}, i_t)^*, (i_{t+3}, i_{t+2})^*, \dots, (i_{j-1}, i_{j-2})^*\}$ if j is odd.

If $j \in \{3, \dots, t-1\}$, j must be even (since t is odd, and $\alpha_{i_t i_j} = 1$ for all odd j). Then we set $A^* = \{(i_{t+2}, i_{t+3}), \dots, (i_k, i_1), (i_1, i_2), \dots, (i_{j-2}, i_{j-1})\} \cup \{(i_{t+1}, i_t)^*, (i_t, i_j), (i_j, i_{j-1})^*\} \cup \{(i_{j+2}, i_{j+1})^*, (i_{j+4}, i_{j+3})^*, \dots, (i_{t-1}, i_{t-2})^*\}$.

Finally, if $j = 1$, we must have $t \neq k$, and we set $A^* = \{(i_t, i_1), (i_1, i_2), \dots, (i_{t-2}, i_{t-1})\} \cup \{(i_{t+1}, i_t)^*, (i_{t+3}, i_{t+2})^*, \dots, (i_k, i_{k-1})^*\}$ if k is even; and $A^* = \{(i_t, i_1)\} \cup \{(i_1, i_k)^*, (i_{k-1}, i_{k-2})^*, \dots, (i_4, i_3)^*\}$ if k is odd.

(c) We prove the validity of $\alpha x \leq \alpha_0$ by giving its Chvátal derivation. All the nonzero multipliers of this derivation can be expressed in terms of the constant

$$\mu := \begin{cases} 2^{k/2} - 1 & \text{if } k (=|C|) \text{ is even} \\ 2^{(k-1)/2} & \text{otherwise.} \end{cases}$$

Here is the list of inequalities to be combined, along with their multipliers:

- the subtour elimination constraint (SEC) associated with $N(C)$, with multiplier $\frac{1}{\mu}$;
- the SEC associated with $N(C_{i_1 i_j})$, for j odd, $3 \leq j \leq k$, with multiplier $\frac{2^{(j-3)/2}}{\mu}$;
- the SEC associated with $N(C_{i_{j+1} i_j})$, for j odd, $3 \leq j \leq k-1$, with multiplier $\frac{\mu-2^{(j-1)/2}}{\mu}$;
- the outdegree inequality associated with node i_j for j even, $4 \leq j \leq k$, with multiplier $\frac{2^{(j-2)/2}-1}{\mu}$;
- the indegree inequality associated with node i_j for j odd, $3 \leq j \leq k$, with multiplier $\frac{\mu-2^{(j-1)/2}}{\mu}$;
- the outdegree inequality associated with node i_1 , with multiplier $\frac{\mu-1}{\mu}$.

It is not hard to check that the above combination of inequalities, with coefficients rounded down, yields $\alpha x \leq \alpha_0$. Figure 8 is meant to illustrate the procedure for the case of k ($= |C|$) = 12 (even) and 11 (odd). For the SEC associated with $N(C_{i_a i_b})$, the corresponding multiplier appears on the 2-chord (i_a, i_b) ; whereas for the SEC associated with $N(C)$, it appears on the arc (i_1, i_2) of C . For each outdegree inequality associated with a node i_j (marked with +), the multiplier appears near the node; the same holds for each indegree inequality associated with a node i_j (marked with -). \square

Theorem 4.4 *Let $w, a_1, \dots, a_{k_a}, b_1, \dots, b_{k_b}$, be distinct nodes with $k_a, k_b \geq 1$, $k_b \in \{k_a, k_a + 1\}$, and $k_a + k_b + 1 \leq n - 1$. Let C be the directed cycle visiting, in sequence, nodes $w, b_1, b_2, \dots, b_{k_b}, a_{k_a}, a_{k_a-1}, \dots, a_1, w$, and define*

$$F := \bigcup_{i=1}^{k_a} \{(a_i, b_j) : i \leq j \leq i+1, j \leq k_b\}.$$

Then the fork inequality

$$\alpha x := x(C) + x(F) + \sum_{i=1}^{k_a} \sum_{j=1}^{k_b} x_{a_i b_j} + \sum_{i=1}^{k_a-1} x(a_i, \{a_{i+1}, \dots, a_{k_a}\}) + \sum_{j=1}^{k_b-1} x(\{b_{j+1}, \dots, b_{k_b}\}, b_j) \leq k_a + k_b =: \alpha_0$$

is a rank 1 maximally 2-lifted cycle inequality, hence facet defining for \tilde{P} (see Figure 9).

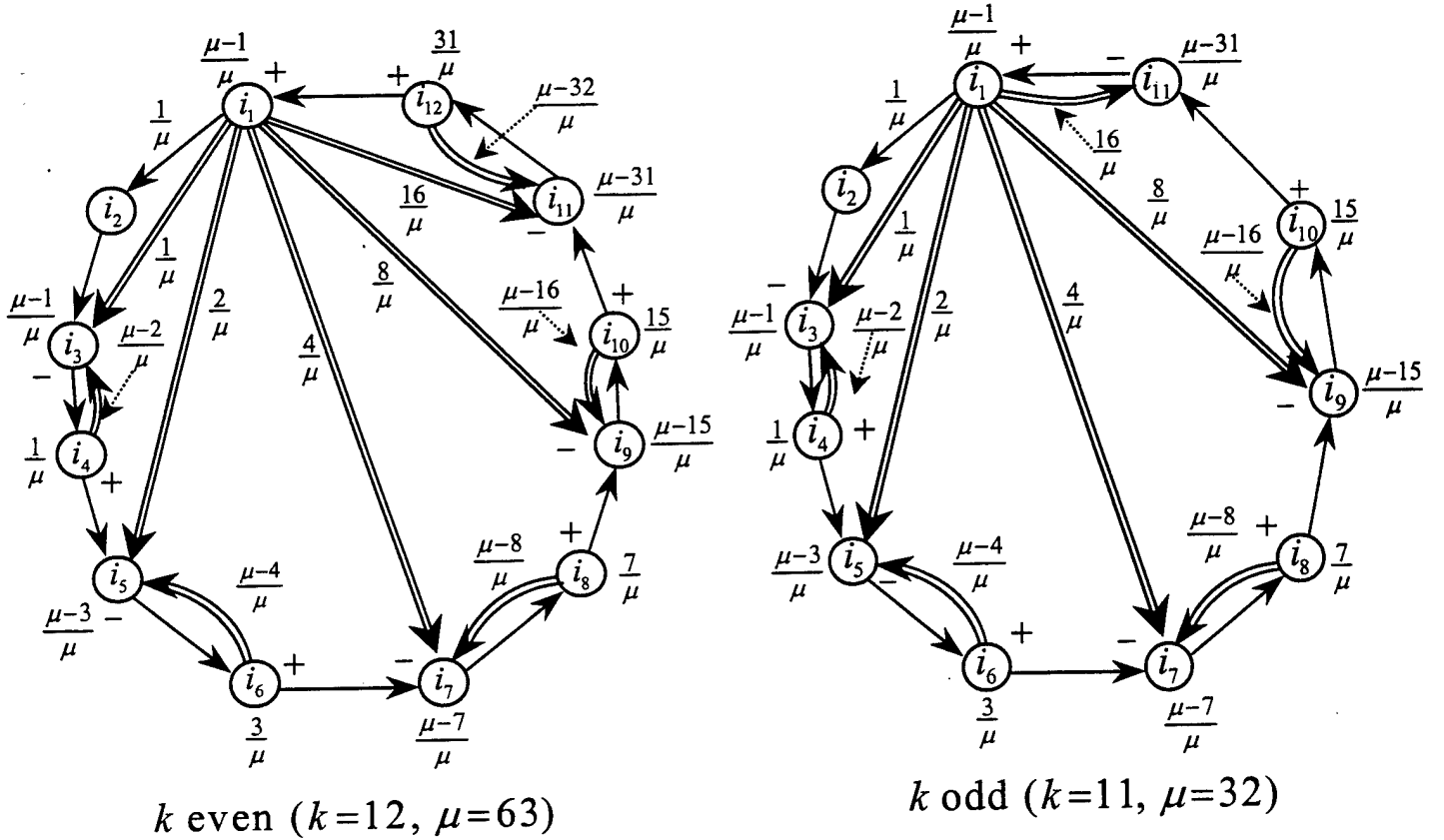


Figure 8: Illustration of the Chvátal derivation of a shell inequality.

Proof. We use the same reasoning as for Theorem 4.3, i.e. we show that the fork inequality satisfies conditions (a), (b), (c) of Theorem 3.2. Notice that F is the set Q_2 of chords with coefficient 2.

(a) Same as in Theorem 4.3.

(b) Let (u, v) be such that $\alpha_{uv} = 0$. We claim that α_{uv} is maximal. If $w \in \{u, v\}$, this follows from Corollary 2.3. Otherwise there are two cases to be considered.

Case 1. $u = a_i$ for some $i \in \{1, \dots, k_a\}$. Then $v = a_j$ for some $j \leq i - 2$, and the claim follows from Theorem 2.1, case (i), as it applies to the 2-chord (a_{j+1}, b_{j+1}) .

Case 2. $u = b_j$ for some $j \in \{1, \dots, k_b\}$. If $v = b_i$ for some $i \geq j + 2$, the claim follows from Theorem 2.1, case (ii), as it applies to the 2-chord (a_{j+1}, b_{j+1}) . Otherwise $v = a_i$ for some $i \in \{1, \dots, k_a\}$, and the claim is proved by exhibiting $x^* \in \tilde{P}$ with support $A^* := \{(a_{i-1}, a_{i-2}), (a_{i-2}, a_{i-3}), \dots, (a_2, a_1), (a_1, w), (w, b_1), (b_1, b_2), \dots, (b_{j-1}, b_j), (b_j, a_i), (a_i, a_{i+1}), \dots, (a_{k_a-1}, a_{k_a}), (a_{k_a}, b_{k_b})^*, (b_{k_b}, b_{k_b-1}), \dots, (b_{j+2}, b_{j+1})\}$.

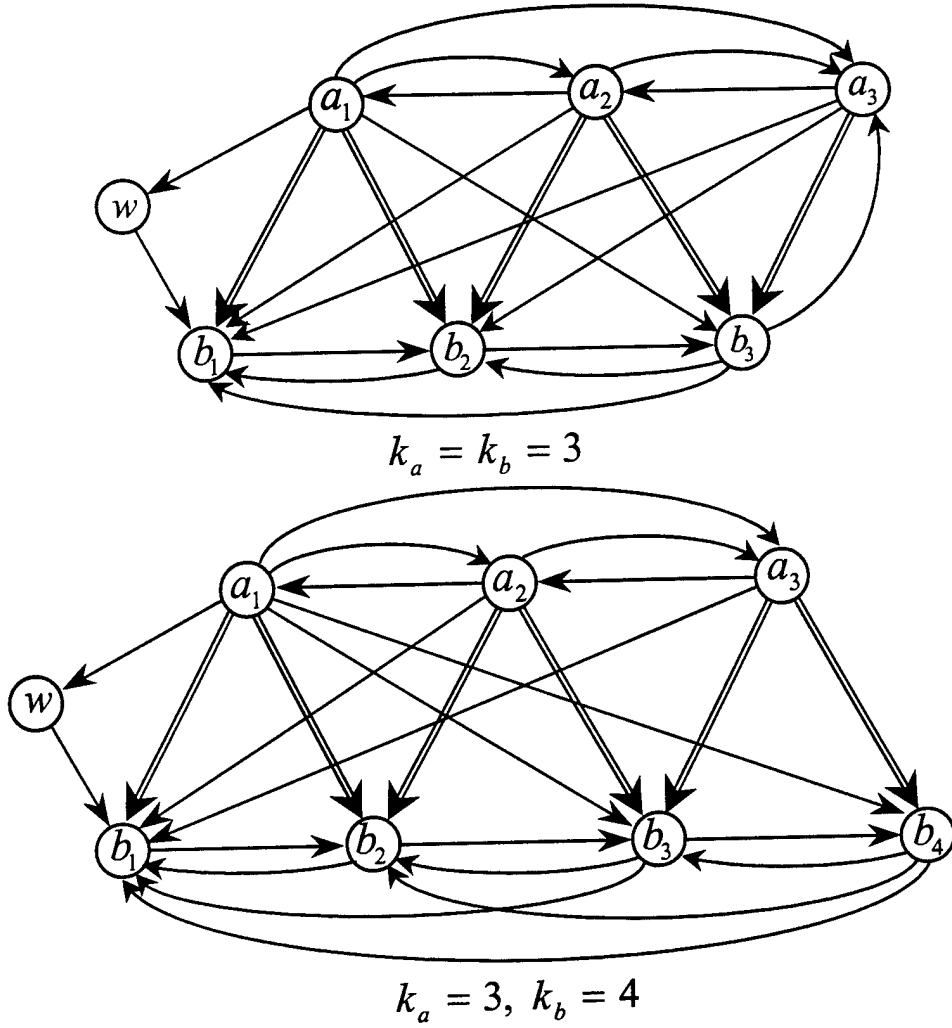


Figure 9: Support graphs of two fork inequalities.

(c) We give the following Chvátal derivation of the fork inequality. For $h = 1, 2, \dots$, we define

$$\sigma_h := \begin{cases} 1 & \text{if } h \in \{1, 2, 3\} \\ \sigma_{h-1} + \sigma_{h-2} & \text{if } h \geq 4. \end{cases}$$

It is easy to see that $\sigma_{h+2} = \sum_{i=1}^h \sigma_i$ for all h . Now let $\mu := \sigma_{|F|+3}$, where F is as in the theorem. Here is the list of inequalities to be combined, along with their multipliers (see Figure 10 for an illustration).

- The SEC associated with $N(C)$, with multiplier $\frac{\sigma_1}{\mu}$;
- For each chord $(a_i, b_j) \in F$, the SEC associated with $N(C_{a_i b_j})$, with multiplier $\frac{\sigma_{i+j}}{\mu}$;
- For each node a_i , $i = 1, \dots, k_a$, the associated outdegree inequality, with multiplier $\frac{\mu - \sigma_{2i+1}}{\mu}$;

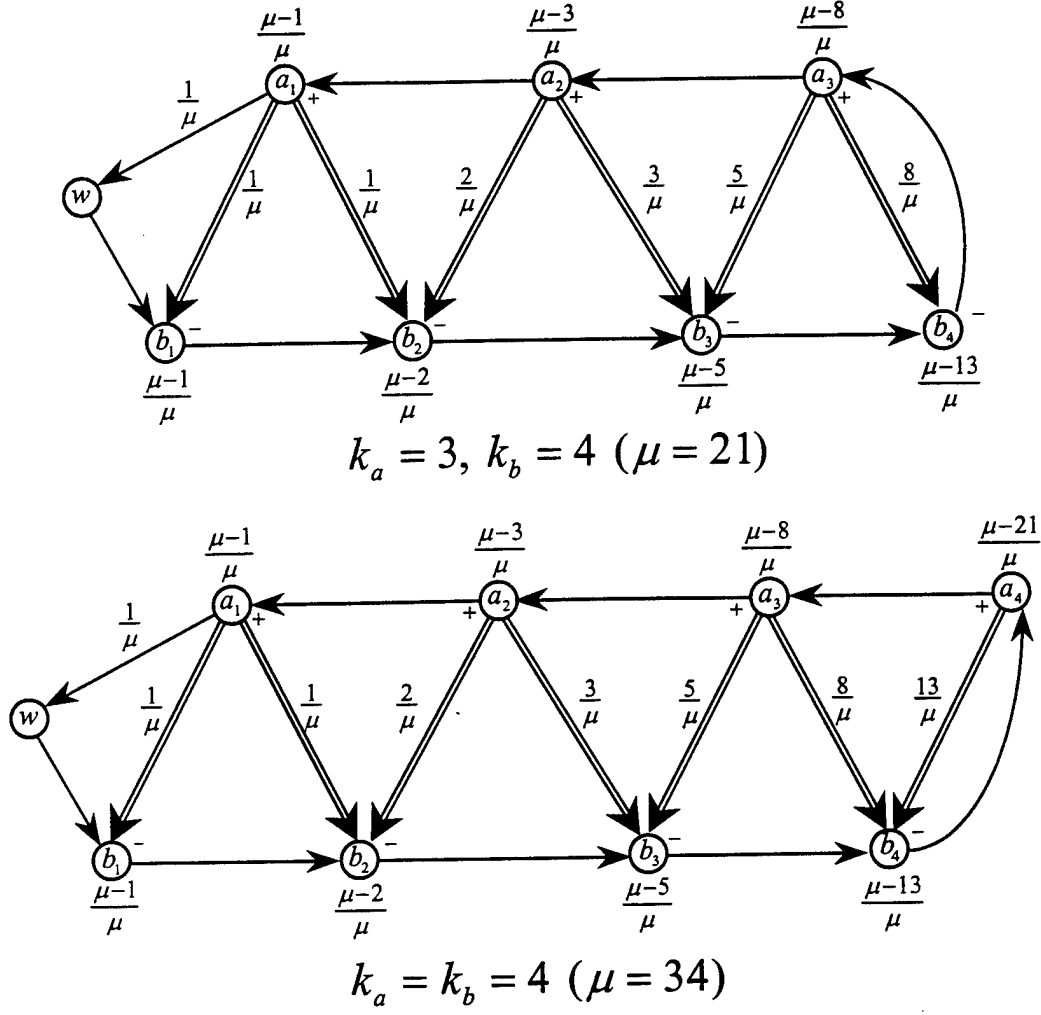


Figure 10: Illustration of the Chvátal derivation of a fork inequality.

- For each node b_j , $j = 1, \dots, k_b$, the associated indegree inequality, with multiplier $\frac{\mu - \sigma_{2j}}{\mu}$.

5 Maximally 2-Lifted Cycle Inequalities of Unbounded Rank

We next establish an important property of the family of lifted cycle inequalities.

Theorem 5.1 *The family of lifted cycle inequalities contains members of unbounded Chvátal rank.*

Proof. Let C be a cycle with node set i_1, \dots, i_k , $k \geq 8$ even, and consider the nonmaximal 2-liftable chord set $Q := \{(i_1, i_3), (i_3, i_5), \dots, (i_{k-1}, i_1)\}$. Let $\alpha x \leq \alpha_0$ be any lifted cycle inequality associated with C and such that $\alpha_{ij} = 2$ for all $(i, j) \in Q$. We claim that for any subset S of the set

$N_{\text{even}} := \{i_2, i_4, \dots, i_k\}$ of even nodes, the subtour elimination inequality associated with $N(C) \setminus S$ must have a positive multiplier in any Chvátal derivation of $\alpha x \leq \alpha_0$. Since the number of subsets S is exponential in k (which in turn is bounded only by n), it then follows from Chvátal, Cook and Hartmann [7] that for any $M > 0$ there exists a lifted cycle inequality in a sufficiently large digraph, whose Chvátal rank is at least M .

To prove the claim, we use the same argument as in the proof of Lemma 4.2. Namely, suppose there exists a Chvátal derivation of $\alpha x \leq \alpha_0$ in which the SEC associated with some $S^* := N(C) \setminus S$ with $S \subseteq N_{\text{even}}$ has 0 multiplier. Then $\alpha x \leq \alpha_0$ must be valid for the polytope \tilde{P}^* defined as the convex hull of points $x \in \{0, 1\}^A$ satisfying the degree inequalities and all the SEC's except for the one associated with S^* . But here is a point $x^* \in \tilde{P}^*$ that violates $\alpha x \leq \alpha_0$. To construct the support A^* of x^* , start with $A^* := C$ and then, for each $i_j \in S$, remove from A^* the two arcs incident with i_j (having coefficient 1), and add the arc $(i_{j-1}, i_{j+1}) \in Q$ (which has coefficient 2). Then $\alpha x^* = |C| > \alpha_0$. \square

Next we introduce a large family of maximally 2-lifted cycle inequalities whose set Q_2 of 2-lifted chords is of the type used for the proof of Theorem 5.1, and which therefore contains members with unbounded Chvátal rank. Notice that in addition, these inequalities satisfy the condition of Corollary 3.3.

Theorem 5.2 *Let C be the cycle visiting in sequence the nodes i_1, i_2, \dots, i_{4k} for some integer $k \geq 2$ satisfying $4k \leq n - 1$. Further, let $S_1 := \{i_j \in N(C) : j \text{ is odd}\}$, and $C_1 := \{(i_1, i_3), (i_3, i_5), \dots, (i_{4k-1}, i_1)\}$. Then the curtain inequality*

$$\alpha x := x(C) + x(\gamma(S_1)) + x(C_1) + \sum_{\substack{j=3 \\ j \text{ odd}}}^{2k-1} (x_{i_j i_{4k-j+2}} + x_{i_{4k-j+2} i_j}) \leq 4k - 1 =: \alpha_0$$

is a maximally 2-lifted, hence facet defining, cycle inequality for \tilde{P} .

Proof. Let $Q_t := \{(i, j) : \alpha_{ij} = t\}$ for $t \in \{0, 1, 2\}$. Clearly, Q_2 has no crossing chords and no nooses, so it is 2-liftable. Also, $|Q_2| = |C|/2 + (|C| - 4)/2 = |C| - 2$, hence Q_2 is maximal. Further,

Q_0 is easily seen to consist precisely of those chords whose coefficient is forced to 0 by the conditions of Corollary 2.3. Hence we only have to show that the curtain inequality is valid for \tilde{P} .

From the properties of the 2-liftable chord set Q_2 , the inequality

$$x(C) + 2x(Q_2) \leq 4k - 1$$

is valid for $\tilde{P}(R \setminus Q_2)$. Thus the curtain inequality is satisfied by all $x \in \tilde{P}$ such that $x(Q_1) = 0$. Now let $x \in \tilde{P}$ be such that $x(Q_1) \geq 1$. Then subtracting this inequality from the sum of the $2k$ indegree inequalities and the $2k$ outdegree inequalities for the odd nodes $i_1, i_3, \dots, i_{4k-1}$, produces an inequality $\beta x \leq 4k - 1$, with $\beta \geq \alpha$. This proves the validity of the curtain inequality for \tilde{P} . \square

The pattern of chords with coefficient 2 in a curtain inequality is illustrated in Figure 11, where odd and even nodes are marked by $+$ and $-$, respectively. The chords with coefficient 1 are all those (not shown in the figure) joining pairs of odd nodes.

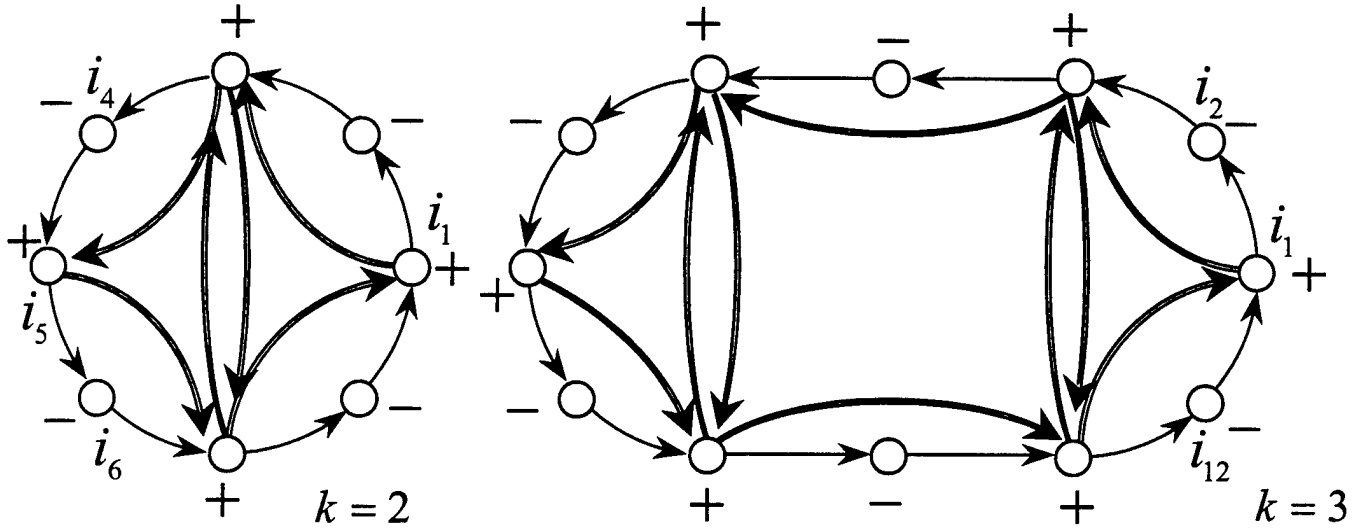


Figure 11: 2-liftable chord sets of curtain inequalities.

The class of curtain inequalities can be extended to cycles of length $|C| \neq 0 \pmod{4}$. The corresponding graphs for $|C| = 5, 6, 7$ are shown in Figure 12.

For the cases $|C| = 1 \pmod{4}$ and $|C| = 3 \pmod{4}$ we have the following.

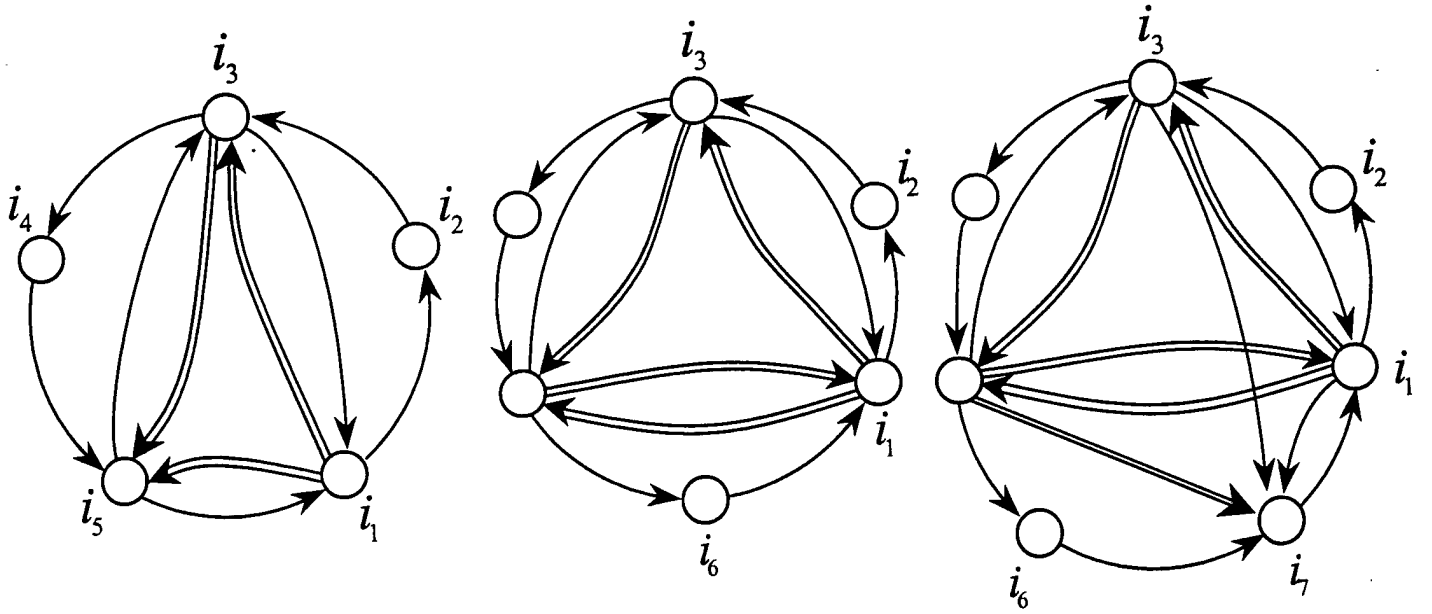


Figure 12: Support graphs of curtain inequalities for $|C| = 5, 6, 7$.

Theorem 5.3 Let C be the cycle visiting in sequence the nodes i_1, \dots, i_{4k+1} for some integer k , $2 \leq k < n/4$. Further, let $S_1 := \{i_j \in N(C) : j \text{ is odd}\}$, and $P_1 := \{(i_1, i_3), (i_3, i_5), \dots, (i_{4k-1}, i_{4k+1})\}$. Then the curtain inequality

$$\alpha x := x(C) + x(\gamma(S_1)) + x(P_1) + \sum_{\substack{j=3 \\ j \text{ odd}}}^{2k-1} (x_{i_j i_{4k-j+2}} + x_{i_{4k-j+2} i_j}) + x_{i_1 i_{4k+1}} \leq 4k =: \alpha_0$$

is a maximally 2-lifted, hence facet defining, inequality for \tilde{P} .

Proof. Parallels the proof of Theorem 5.2. As in that case, we only have to show that the inequality is satisfied by all $x \in \tilde{P}$ such that $x(Q_1) \geq 1$. Let x have this property. Then adding

- the outdegree inequalities for nodes $i_1, i_3, \dots, i_{4k-1}$
- the indegree inequalities for node $i_3, i_5, \dots, i_{4k+1}$
- $1/2$ times the outdegree inequality for node i_{4k+1}
- $1/2$ times the indegree inequality for node i_1
- $1/2$ times the inequality $-x(Q_1) \leq -1$

we obtain an inequality $\beta x \leq 4k + 0.5$, with $\beta \geq \alpha$. Rounding down the coefficients on both sides then yields an inequality that implies $\alpha x \leq 4k$. \square

Theorem 5.4 *Let C be the cycle visiting in sequence the nodes i_1, \dots, i_{4k+3} for some integer k , $2 \leq k \leq n/4$. Further, let $S_1 := \{i_j \in N(C) : j \text{ is odd}\}$, and $P_1 := \{(i_1, i_3), (i_3, i_5), \dots, (i_{4k+1}, i_{4k+3})\}$. Then the curtain inequality*

$$\alpha x := x(C) + x(\gamma(S_1)) + x(P_1) + \sum_{\substack{j=1 \\ j \text{ odd}}}^{2k-1} (x_{i_j i_{4k-j+2}} + x_{i_{4k-j+2} i_j}) - \sum_{\substack{j=3 \\ j \text{ odd}}}^{4k+1} x_{i_{4k+3} i_j} \leq 4k + 2 := \alpha_0$$

is a maximally 2-lifted, hence facet defining, inequality for \tilde{P} .

Proof. As in the case of Theorem 5.2, Q_2 can easily be seen to be a maximal 2-liftable chord set. Also, from Theorem 2.2, all chords incident from or to even nodes have 0 coefficients. Further, from the same theorem as it applies to the 2-chord (i_{4k+1}, i_1) , all chords incident from i_{4k+3} have 0 coefficients. Finally, to prove validity, the inequality can be shown to be satisfied by all $x \in \tilde{P}$ such that $x(Q_1) \geq 1$ by adding

- the outdegree inequalities for nodes $i_1, i_3, \dots, i_{4k+1}$
- the indegree inequalities for nodes $i_1, i_3, \dots, i_{4k+3}$
- the inequality $-x(Q_1) \leq -1$. \square

Finally, for the case $|C| \equiv 2 \pmod{4}$ we have a stronger result, i.e. we can identify a larger class of facet defining inequalities that contains as a special case the curtain inequality with $|C| \equiv 2 \pmod{4}$.

Theorem 5.5 *Let C be an even length cycle visiting nodes $i_1, \dots, i_{|C|}$. Define the cycle*

$$C_1 := \{(i_1, i_3), (i_3, i_5), \dots, (i_{|C|-1}, i_1)\}$$

and let S_1 be the node set of C_1 . Then for any maximally 2-liftable chord set Q_2 containing C_1 , the inequality

$$\alpha x := x(C) + x(\gamma(S_1)) + x(Q_2) \leq |C| - 1$$

is a maximally 2-lifted, hence facet defining, inequality for \tilde{P} .

Proof. Since Q_2 is maximally 2-liftable, every $x \in P$ satisfies $x(C) + x(Q_2) \leq |C| - 1$. Thus we only need to prove that the inequality of the Theorem is valid for \tilde{P} . Clearly, all $x \in \tilde{P}$ such that $x(Q_1) = 0$, where Q_1 is the set of chords with coefficient 1, satisfies the inequality. Now let $x \in \tilde{P}$ be such that $x(Q_1) \geq 1$, and note that $|S_1| = |C|/2$. Then adding up

- the outdegree inequalities for nodes $i \in S_1$
- the indegree inequalities for nodes $i \in S_1$
- the inequality $-x(Q_1) \leq -1$,

we obtain an inequality $\beta x \leq |C| - 1$, where $\beta \geq \alpha$. \square

The curtain inequality for $|C| = 2 \pmod{4}$ is then a special case of the inequality of Theorem 5.5.

Corollary 5.6 *Let C be the cycle visiting in sequence the nodes $i_1, i_2, \dots, i_{4k+2}$ for some integer $k \geq 2$ satisfying $4k + 2 \leq n - 1$. Further, let $S_1 := \{i_j \in N(C) : j \text{ is odd}\}$, and $C_1 := \{(i_1, i_3), (i_3, i_5), \dots, (i_{4k+1}, i_1)\}$. Then the curtain inequality*

$$\alpha x := x(C) + x(\gamma(S_1)) + x(C_1) + \sum_{\substack{j=3 \\ j \text{ odd}}}^{2k-1} (x_{i_j i_{4k-j+2}} + x_{i_{4k-j+2} i_j}) + x_{i_1 i_{4k+1}} \leq 4k + 1 =: \alpha_0$$

is a maximally 2-lifted, hence facet defining, cycle inequality for \tilde{P} .

6 Separation

The structure of the lifted cycle inequalities lends itself to relatively easy (heuristic) separation procedures. Here is an example of a polynomial time separation procedure that works in many, though not all cases. Whenever the condition of the following Theorem is satisfied, which can be checked in $O(n^3)$ time, one can derive at least one, and sometimes several, violated valid inequalities.

Theorem 6.1 *Let \bar{x} be a fractional point of the subtour elimination polytope, such that*

$$\bar{x}(\gamma(S)) = |S| - 1 \quad (1)$$

for some $S \subset N$, $2 \leq |S| \leq n - 2$. If there exists a collection of distinct arcs $(i_r, j_r) \in \gamma(S)$ and associated distinct nodes $k_r \in N \setminus S$, $r = 1, \dots, q$, where $1 \leq q \leq n - |S| - 1$, such that

$$\bar{x}_{i_r j_r} + \bar{x}_{i_r k_r} + \bar{x}_{k_r j_r} > 1, \quad r = 1, \dots, q, \quad (2)$$

then the inequality

$$x(\gamma(S)) + \sum_{r=1}^q (x_{i_r j_r} + x_{i_r k_r} + x_{k_r j_r}) \leq |S| + q - 1 \quad (3)$$

is valid for the ATS polytope and violated by \bar{x} .

Proof. We show that (3) is valid by induction on q , where at each step of the induction we use a Chvatal derivation. For $q = 1$, we add

$$\begin{aligned} & \frac{1}{2} \text{ times the inequality } x(\gamma(S)) \leq |S| - 1, \\ & \frac{1}{2} \text{ times the inequality } x(\gamma(S \cup \{k_1\})) \leq |S|, \text{ and} \\ & \frac{1}{2} \text{ times the inequality } 2x_{i_1 j_1} + x_{i_1 k_1} + x_{k_1 j_1} \leq 2 \end{aligned}$$

to obtain, after rounding down the resulting inequality,

$$x(\gamma(S)) + x_{i_1 j_1} + x_{i_1 k_1} + x_{k_1 j_1} \leq |S|,$$

which is the special case of (3) for $q = 1$. Suppose (3) is valid for $q = 1, \dots, t - 1$, and let $q = t$.

Now add

$$\begin{aligned} & \frac{1}{2} \text{ times the inequality } x(\gamma(S)) + \sum_{r=1}^{t-1} (x_{i_r j_r} + x_{i_r k_r} + x_{k_r j_r}) \leq |S| + t - 2 \\ & \frac{1}{2} \text{ times the inequality } x(\gamma(S \cup \{k_t\})) + \sum_{r=1}^{t-1} (x_{i_r j_r} + x_{i_r k_r} + x_{k_r j_r}) \leq |S| + t - 1, \end{aligned}$$

(both inequalities being valid by the induction hypothesis), and

$$\frac{1}{2} \text{ times the inequality } 2x_{i_t j_t} + x_{i_t k_t} + x_{k_t j_t} \leq 2,$$

to obtain, after rounding down the resulting inequality,

$$x(\gamma(S)) + \sum_{r=1}^t (x_{i_r j_r} + x_{i_r k_r} + x_{k_r j_r}) \leq |S| + t - 1,$$

thus proving that (3) is valid.

To show that (3) is violated by \bar{x} , notice that from (1) and (2) we have

$$\bar{x}(\gamma(S)) + \sum_{r=1}^q (\bar{x}_{i_r j_r} + \bar{x}_{i_r k_r} + \bar{x}_{k_r j_r}) > |S| - 1 + q. \square$$

The inequality (3) can also be obtained through V -lifting, an operation introduced by Ascheuer, Fischetti and Grötschel [1], in the context of the TSP with time windows.

A few comments are in order on the nature of the inequality (3). If the arcs (i_r, j_r) , $r = 1, \dots, q$ are pairwise compatible and do not form a cycle shorter than $|S|$, then inequality (3) is obtainable from a cycle inequality $x(C) \leq |C| - 1$ through sequential but not necessarily maximal lifting. Here C is obtained from any directed cycle C' with node set S and containing the arcs (i_r, j_r) , $i = 1, \dots, q$, by replacing every arc (i_r, j_r) with the pair of arcs $(i_r, k_r), (k_r, j_r)$. The arcs (i_r, j_r) , $r = 1, \dots, q$, then form a (not necessarily maximal) 2-liftable chord set with respect to the cycle C .

As to the complexity of the separation procedure for identifying inequalities of class (3), if the set S for which condition (1) holds is given (which is the case whenever the separation procedure is applied after the separation of subtour elimination inequalities), then examining every arc $(i_r, j_r) \in \gamma(S)$ and trying to identify a node $k_r \in N \setminus S$ such that (i_r, j_r) and k_r satisfy condition (2), takes at most $|S|^2(n - |S|)$ steps, i.e. the procedure is $O(n^3)$. In case the set S is not given, the above separation procedure is still polynomial in n , as the number of subtour elimination constraints satisfied at equality by any point \bar{x} of the subtour elimination polytope is known to be $O(n^2)$, and their identification requires $O(n^3 \log n)$ steps [12].

Example. Consider the graph on 8 nodes with the feasible point \bar{x} shown in Figure 13 (\bar{x} satisfies all the subtour elimination inequalities on 2 to 7 nodes). The subtour elimination inequality on $S := \{5, 6, 7, 8\}$ is satisfied by \bar{x} at equality: $\bar{x}_{56} + \bar{x}_{67} + \bar{x}_{78} + \bar{x}_{85} = 3$. The arc set $\{(5, 6), (6, 7), (7, 8)\}$

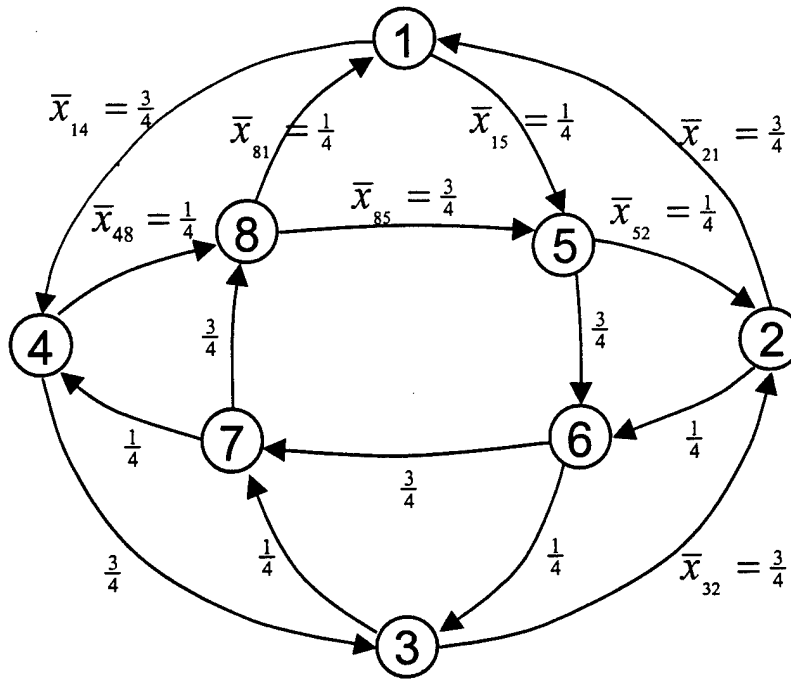


Figure 13: Feasible point \bar{x} of the subtour elimination polytope.

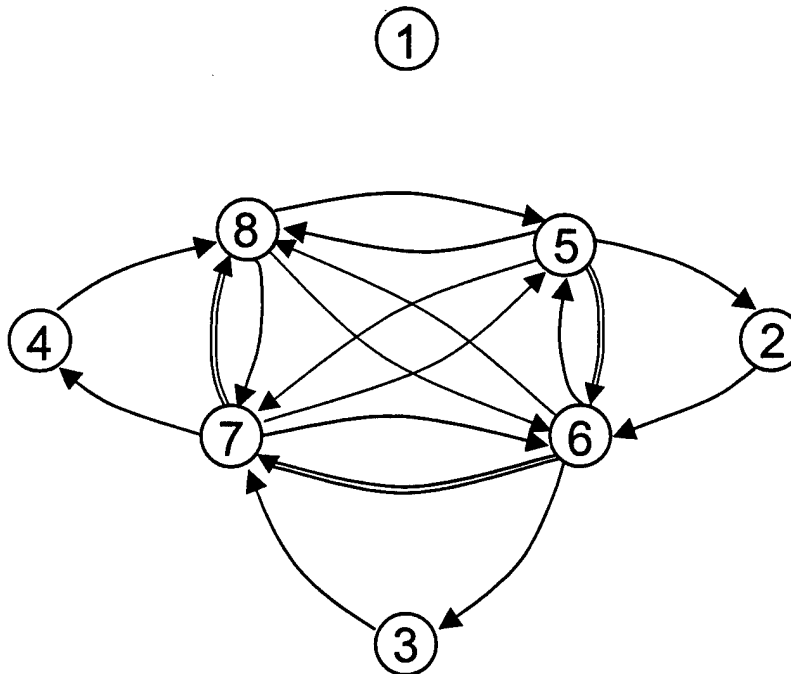


Figure 14: Support of 2-lifted cycle inequality violated by \bar{x} .

and associated node set $\{2, 3, 4\}$ satisfy the requirement of the theorem: $\bar{x}_{56} + \bar{x}_{52} + \bar{x}_{26} = \bar{x}_{67} + \bar{x}_{63} + \bar{x}_{37} = \bar{x}_{78} + \bar{x}_{74} + \bar{x}_{48} = \frac{5}{4} > 1$. Thus the inequality

$$x(\gamma(\{5, 6, 7, 8\})) + (x_{56} + x_{52} + x_{26}) + (x_{67} + x_{63} + x_{37}) + (x_{78} + x_{74} + x_{48}) \leq 4 + 3 - 1 = 6,$$

whose support is shown in Figure 14, is valid but violated by \bar{x} , for which the lefthand side takes on the value $\frac{27}{4} > 6$.

Starting with the same set S , but using any nonempty subset of size ≤ 3 of the arc set $\{(5, 6), (6, 7), (7, 8), (8, 5)\}$, we obtain a different lifted cycle inequality violated by \bar{x} . Furthermore, using as a starting set $S' := \{1, 2, 3, 4\}$, we obtain another set of inequalities violated by \bar{x} .

7 New Inequalities for the Symmetric TS Polytope

It is well known that every ATS problem defined on the complete digraph $G = (N, A)$ can be restated as a STS problem defined on a special undirected graph $G^* := (V^*, E^*)$, where V^* has a pair of vertices i^+, i^- for every node $i \in N$, and E^* has an edge (i^+, j^-) for every arc $(i, j) \in A$ and an edge (i^+, i^-) for every node $i \in N$, with the condition that the only feasible tours in G^* are those that contain every edge (i^+, i^-) , $i = 1, \dots, n$. This equivalence can be used to find new facets of the (general) STS problem as follows.

Let $\alpha x \leq \alpha_0$ be any asymmetric facet defining inequality for the ATS polytope. To every incidence vector $x \in \{0, 1\}^A$ of an arc set in G associate the incidence vector $y \in \{0, 1\}^{E^*}$ of an edge set in G^* , as follows: $y_{i+j-} = x_{ij}$ for all $(i, j) \in A$, and $y_{i+i-} = 1$ for all $i \in N$. Also, associate to the vector $\alpha \in \mathbb{R}^A$ a vector $\beta \in \mathbb{R}^{E^*}$ by the rule that $\beta_{i+j-} = \alpha_{ij}$ for all $(i, j) \in A$, and $\beta_{i+i-} = 0$ for all $i \in N$. Then the inequality $\beta y \leq \alpha_0$ defines a facet of the face of the STS polytope defined on the complete graph \tilde{G}^* with vertex set V^* by the equations $y_{i+i-} = 1$, $i = 1, \dots, n$, and $y_{i+j+} = y_{i-j-} = 0$ for all $i, j = 1, \dots, n$, $i \neq j$. One can then free the variables fixed at 1 or 0 and calculate lifting coefficients for them. The outcome is a facet defining inequality $\beta y \leq \beta_0$ for the STS polytope on the complete graph \tilde{G}^* . Sometimes the resulting inequalities are of a known

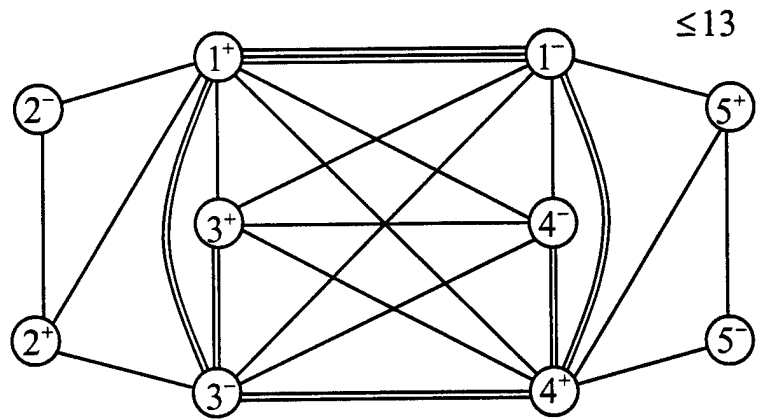
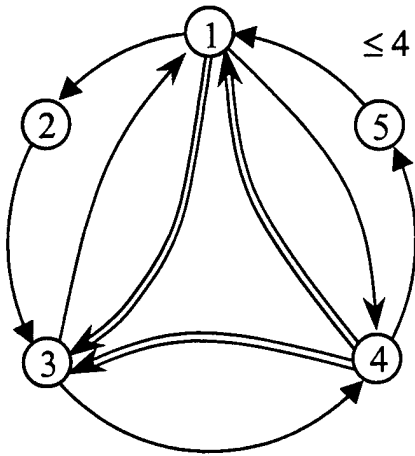


Figure 15: Curtain inequality for the ATSP and corresponding inequality for the STSP.

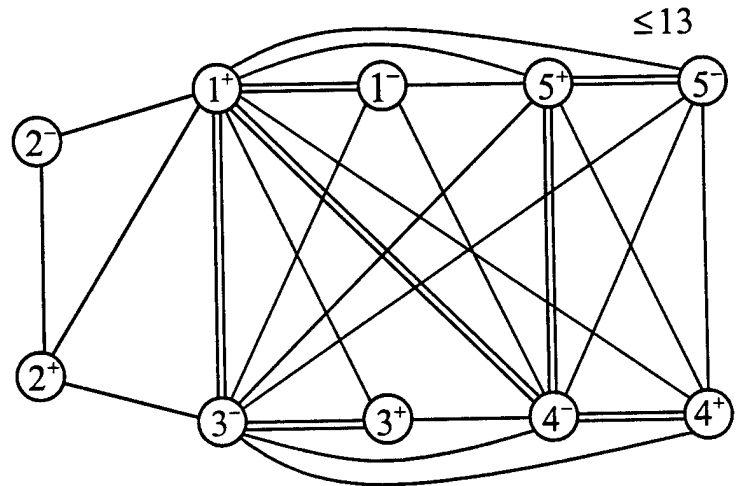
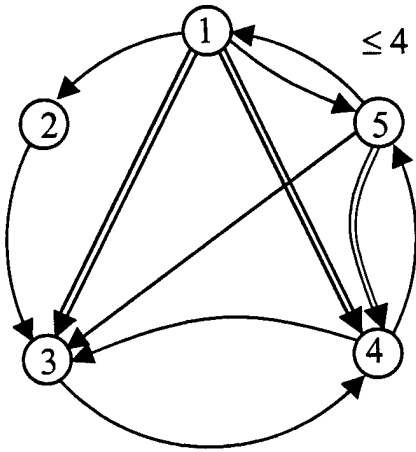


Figure 16: Fork inequality for the ATSP and corresponding inequality for the STSP.

type; but when $\alpha x \leq \alpha_0$ is a lifted cycle inequality then the corresponding inequality $\beta x \leq \beta_0$ for the STS polytope on the complete graph often seems to be new. Figures 15 and 16 show two lifted cycle inequalities for the ATS polytope and their counterparts for the STS polytope. We are currently investigating more closely the polyhedral implications of the transformation of an ATSP instance to a STSP instance. Apparently, this issue has not been considered before.

References

- [1] N. Ascheuer, M. Fischetti and M. Grötschel, "A Polyhedral Study of the Asymmetric Traveling Salesman Problem with Time Windows." Technical Report, ZIB, Berlin, February 1997.
- [2] E. Balas, "The Asymmetric Assignment Problem and Some New Facets of the Traveling Salesman Polytope." *SIAM Journal on Discrete Mathematics*, 2 (1989), 425-451.
- [3] E. Balas and M. Fischetti, "The Fixed-Outdegree 1- Arborescence Polytope." *Mathematics of Operations Research*, 17 (1992), 1001-1018.
- [4] E. Balas and M. Fischetti, "A Lifting Procedure for the Asymmetric Traveling Salesman Polytope and a Large New Class of Facets." *Mathematical Programming*, 58, 1993, 325-352.
- [5] E. Balas and M. Fischetti, "On the Monotonization of Polyhedra." In G. Rinaldi and L. Wolsey (editors), *Integer Programming and Combinatorial Optimization* (Proceedings of IPCO 3), 1993, 23-38.
- [6] S. Chopra and G. Rinaldi, "The Graphical Asymmetric Traveling Salesman Polyhedron." In R. Kannan and W. Pulleyblank (editors), *Integer Programming and Combinatorial Optimization* (Proceedings of IPCO 1), 1990, 129-146.
- [7] V. Chvátal, W. Cook and M. Hartmann, "On Cutting Plane Proofs in Combinatorial Optimization." *Linear Algebra and Its Applications*, 114/115, 1989, 455-499.
- [8] M. Fischetti, "Facets of the Asymmetric Traveling Salesman Polytope." *Mathematics of Operations Research*, 16, 1990, 129-146.
- [9] M. Fischetti, "Three Lifting Theorems for the Asymmetric Traveling Salesman Polytope." In E. Balas, G. Cornuéjols and R. Kannan (editors), *Integer Programming and Combinatorial Optimization* (Proceedings of IPCO 2), 1992, 260- 273.
- [10] M. Grötschel and M.W. Padberg, "Lineare Charakterisierungen von Travelling Salesman Problemen." *Zeitschrift Für Operations Research*, 21, 1977, 33-64.
- [11] M. Grötschel and M.W. Padberg, "Polyhedral Theory." In E. Lawler, J.K. Lenstra, A.H.G. Rinnooy Kan and D. Shmoys (editors), *The Traveling Salesman Problem: A Guided Tour to Combinatorial Optimization*. Wiley, 1985, 251-306.
- [12] J. Hao and J. Orlin, "A Faster Algorithm for Finding the Minimum Cut in a Graph." In *Proceedings of the 3rd ACM-SIAM Symposium on Discrete Algorithms*, 1992, p. 165-174.
- [13] M. Jünger, G. Reinelt and G. Rinaldi, "The Traveling Salesman Problem." In M.O. Ball, T.L. Magnanti, C.L. Monma and G.L. Nemhauser (editors), *Network Models*. North-Holland, 1995, 225-330.

- [14] R. Karp, "Reducibility Among Combinatorial Problems." In R.E. Miller and J.W. Thatcher (editors), *Complexity of Computer Computations*. Plenum Press, 1972, 85-103.
- [15] M. Queyranne and Y. Wang, "Symmetric Inequalities and Their Composition for Asymmetric Traveling Salesman Polytopes." University of British Columbia, W.P 90-MSC-002, 1990; revised 1994.
- [16] M. Queyranne and Y. Wang, "On the Chvatal Rank of Certain Inequalities." University of British Columbia, Vancouver, BC, 1994.
- [17] A. Schrijver, *Theory of Linear and Integer Programming*. Wiley, 1986.

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number) We investigate the family of facet defining inequalities for the asymmetric traveling salesman (ATS) polytope obtainable by lifting the cycle inequalities. We establish several properties of this family that earmark it as the most important among the asymmetric inequalities for the ATS polytope known to date: (i) The family is shown to contain members of unbounded Chvatal rank, whereas most known asymmetric inequalities are of Chvatal rank 1. (ii) For large classes within the family a coefficient pattern is identified that makes it easy to develop efficient separation routines. (iii) Each member of the family is shown to have a counterpart for the symmetric TS (STS) polytope that is often new, and is obtainable by mapping the inequality for the ATS polytope into a certain face of the STS polytope and then lifting the resulting inequality into one for the STS polytope itself.		